Please check the examination details below before entering your candidate information

Candidate surname	Other names

Centre Number | Candidate Number

Pearson Edexcel International Advanced Level

Thursday 23 May 2024

Morning (Time: 1 hour 30 minutes) | Paper reference **WFM01/01**

Mathematics

International Advanced Subsidiary/ Advanced Level
Further Pure Mathematics F1

You must have:
Mathematical Formulae and Statistical Tables (Yellow), calculator

Total Marks

Candidates may use any calculator allowed by Pearson regulations. Calculators must not have the facility for symbolic algebra manipulation, differentiation and integration, or have retrievable mathematical formulae stored in them.

Instructions

- Use **black** ink or ball-point pen.
- If pencil is used for diagrams/sketches/graphs it must be dark (HB or B).
- **Fill in the boxes** at the top of this page with your name, centre number and candidate number.
- Answer **all** questions and ensure that your answers to parts of questions are clearly labelled.
- Answer the questions in the spaces provided – *there may be more space than you need.*
- You should show sufficient working to make your methods clear. Answers without working may not gain full credit.
- Inexact answers should be given to three significant figures unless otherwise stated.

Information

- A booklet 'Mathematical Formulae and Statistical Tables' is provided.
- There are 9 questions in this question paper. The total mark for this paper is 75.
- The marks for **each** question are shown in brackets – *use this as a guide as to how much time to spend on each question.*

Advice

- Read each question carefully before you start to answer it.
- Try to answer every question.
- Check your answers if you have time at the end.
- If you change your mind about an answer, cross it out and put your new answer and any working underneath.

Turn over

©2024 Pearson Education Ltd.

1. (i) The matrix **A** is defined by

$$\mathbf{A} = \begin{pmatrix} 3k & 4k-1 \\ 2 & 6 \end{pmatrix}$$

where k is a constant.

(a) Determine the value of k for which **A** is singular.

(2)

Given that **A** is non-singular,

(b) determine $\mathbf{A}^{-1}$ in terms of k, giving your answer in simplest form.

(2)

(ii) The matrix **B** is defined by

$$\mathbf{B} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}$$

where p and q are integers.

State the value of p and the value of q when **B** represents

(a) an enlargement about the origin with scale factor –2

(b) a reflection in the y–axis.

(2)

a. Matrix A is singular $\Rightarrow$ det(A) = 0

$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$\det(X) = (a)(d) - (b)(c)$

$\det(A) = (3k)(6) - (4k-1)(2)$

$18k - (8k-2)$

$18k - 8k + 2$

$10k + 2$

$\det(A) = 10k + 2$

Set $\det(A) = 0 \Rightarrow 10k + 2 = 0$

$10k = -2$

$k = -\frac{2}{10} = -1/5$

$\boxed{k = -\frac{1}{5}}$

P 7 5 7 4 3 A 0 2 2 8

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 1 continued

b. $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $X^{-1} = \dfrac{1}{\det(X)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$-b = -(4k-1) = 1-4k$

$-c = -(2) = -2$

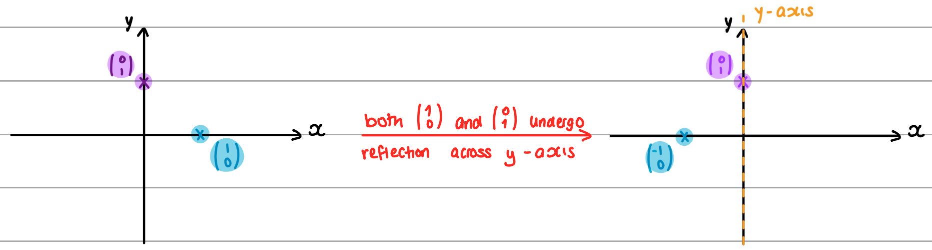
$$\boxed{A^{-1} = \frac{1}{10k+2}\begin{bmatrix} 6 & 1-4k \\ -2 & 3k \end{bmatrix}}$$

ii a. $p = q = -2$

$$\boxed{B = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}}$$

b. Start by drawing a coordinate axis and mark coordinates $\binom{1}{0}$ and $\binom{0}{1}$

then transform both of these coordinates by transformation (reflection in y-axis)



identity matrix I

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{\text{new matrix represents transformation } B} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

rewrite matrix but now replace coordinates with their transformations in this specific order

$$\boxed{B : \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}$$

(Total for Question 1 is 6 marks)

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

2. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

$$f(z) = z^3 - 13z^2 + 59z + p \qquad p \in \mathbb{Z}$$

Given that $z = 3$ is a root of the equation $f(z) = 0$

(a) show that $p = -87$

(2)

(b) Use algebra to determine the other roots of $f(z) = 0$, giving your answers in simplest form.

(4)

On an Argand diagram

- the root $z = 3$ is represented by the point P
- the other roots of $f(z) = 0$ are represented by the points Q and R
- the number $z = -9$ is represented by the point S

(c) Show on a single Argand diagram the positions of P, Q, R and S

(1)

(d) Determine the perimeter of the quadrilateral $PQSR$, giving your answer as a simplified surd.

(2)

a. since $z=3$ is a root, $f(3)=0$

$f(3) = (3)^3 - 13(3)^2 + 59(3) + p = 0$

$27 - 117 + 177 + p = 0$

$87 + p = 0$

$p = -87$

$\boxed{p = -87 \text{ // shown}}$

b. For a cubic eq^n with form: $ax^3 + bx^2 + cx + d$, there are 3 roots.
There are 2 possibilities: 3 real roots
1 real root and 2 complex roots (conjugate pair)

$(z-3)(az^2+bz+c)$

$az^3 + bz^2 + cz - 3az^2 - 3bz + 3c$

$(a)z^3 + (b-3a)z^2 + (c-3b)z + (3c)$

P 7 5 7 4 3 A 0 4 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 2 continued

equate coefficients to $f(z)$: (1) $a=1$

(2) $b-3a=-13$

(3) $c-3b=59$

(4) $3c=p$

(2) $b-3(1)=-13$

$b-3=-13$

$b=-10$

(3) $c-3b=59$

$c-3(-10)=59$

$c+30=59$

$c=29$

quadratic factor, $z^2-10z+29$

$z^2-10z+29=0$

$(z-5)^2-25+29=0$

$(z-5)^2+4=0$

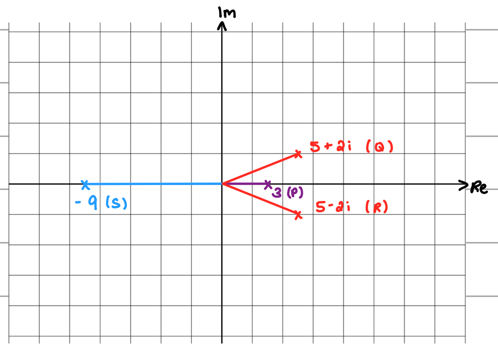
$(z-5)^2=-4$

$(z-5)=\pm\sqrt{-4}$

$z=5\pm 2i$

$\boxed{z=5\pm 2i}$

C.



Question 2 continued

a. find $\|PQ\|$, $\|PR\|$, $\|SQ\|$, $\|SR\|$

$\|PQ\| = \|PR\|$ & $\|SQ\| = \|SR\|$

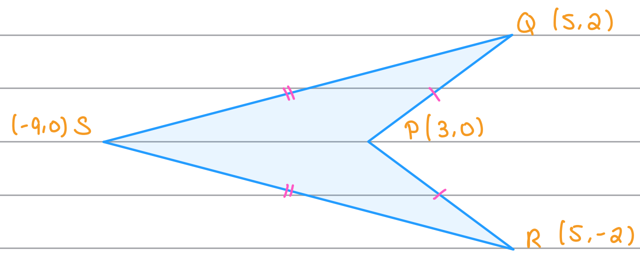
PQRS = $2\|PQ\| + 2\|SQ\|$

Distance between 2 points:
$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$ for points (x_1,y_1) & (x_2,y_2)

$\|PQ\| = \sqrt{(5-3)^2+(2-0)^2} = 2\sqrt{2}$

P(3, 0) Q(5, 2) R(5, -2) S(-9, 0)

$\|SQ\| = \sqrt{(5--9)^2+(2-0)^2} = 10\sqrt{2}$



PQRS = $2(2\sqrt{2}) + 2(10\sqrt{2})$
$= 24\sqrt{2}$

$\boxed{\text{Perimetre PQRS} = 24\sqrt{2} \text{ units}}$

P 7 5 7 4 3 A 0 6 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

3. $$f(x) = x^3 - 5\sqrt{x} - 4x + 7 \qquad x \geqslant 0$$

The equation $f(x) = 0$ has a root α in the interval [0.25, 1]

(a) Use linear interpolation once on the interval [0.25, 1] to determine an approximation to α, giving your answer to 3 decimal places.

(3)

The equation $f(x) = 0$ has another root β in the interval [1.5, 2.5]

(b) Determine $f'(x)$

(2)

(c) Hence, using $x_0 = 1.75$ as a first approximation to β, apply the Newton–Raphson process once to $f(x)$ to determine a second approximation to β, giving your answer to 3 decimal places.

(2)

a. Linear Interpolation:

$\alpha = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ for interval $[a,b]$

where α is the root

$a = 0.25$, $b = 1$

$f(0.25) = (0.25)^3 - 5\sqrt{(0.25)} - 4(0.25) + 7 = \frac{225}{64}$

$f(1) = (1)^3 - 5\sqrt{(1)} - 4(1) + 7 = -1$

$\alpha = \dfrac{0.25 f(1) - 1 f(0.25)}{f(1) - f(0.25)}$

$\alpha = \dfrac{0.25(-1) - 1\left(\frac{225}{64}\right)}{(-1) - \left(\frac{225}{64}\right)}$

$\alpha = 0.8339100346 \quad \left(\frac{241}{289}\right)$

$\boxed{\alpha = 0.834 \quad (3\text{ d.p.})}$

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA

Question 3 continued

b. $f(x) = x^3 - 5\sqrt{x} - 4x + 7$

$f(x) = x^3 - 5x^{\frac{1}{2}} - 4x + 7$

$f'(x) = 3x^2 - \frac{5}{2}x^{-\frac{1}{2}} - 4$

$\boxed{f'(x) = 3x^2 - \frac{5}{2}x^{-\frac{1}{2}} - 4}$

c. $x_1 = 1.75$

we are looking for x_2

Numerical Sol^{n}s of Eq^{n}s:

Newton-Raphson iteration for solving $f(x)=0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

From Newton-Rhapson iteration: $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

(1) must find $f(x_1)$ by subbing in x_1 (1.75) into $f(x)$

(2) must find $f'(x_1)$ by differentiating $f(x)$ and subbing in x_1 (1.75) into $f'(x)$.

(1) $f(1.75) = (1.75)^3 - 5\sqrt{(1.75)} - 4(1.75) + 7 = -1.255003278$

(2) $f'(x) = 3x^2 - \frac{5}{2}x^{-\frac{1}{2}} - 4$ ← use ans from part (b).

$f'(1.75) = 3(1.75)^2 - \frac{5}{2}(1.75)^{-\frac{1}{2}} - 4 = 3.297677635$

$$x_2 = (1.75) - \frac{-1.255003278}{3.297677635}$$

$x_2 = 2.13057185$

$\boxed{\beta = 2.131 \text{ (3 d.p.)}}$

(Total for Question 3 is 7 marks)

4. **In this question you must show all stages of your working.**

Solutions relying entirely on calculator technology are not acceptable.

The complex number z is defined by

$$z = -3 + 4\text{i}$$

(a) Determine $\left|z^2 - 3\right|$

(3)

(b) Express $\dfrac{50}{z^*}$ in the form kz, where k is a positive integer.

(3)

(c) Hence find the value of $\arg\left(\dfrac{50}{z^*}\right)$

Give your answer in radians to 3 significant figures.

(2)

a. $|z^2 - 3| = |(-3+4i)^2 - 3|$

$= |(9 - 24i + 16i^2) - 3|$

$= |9 - 24i + 16(-1) - 3|$

$= |-10 - 24i|$

$= \sqrt{(-10)^2 + (-24)^2}$

$= 26$

$\boxed{|z^2 - 3| = 26}$

b. z^* is the complex conjugate of z.

(flip sign of imaginary component

$z^* = -3 - 4i$

multiply top and bottom by complex conjugate of denominator

$\dfrac{50}{z^*} = \dfrac{50}{-3-4i} \times \dfrac{-3+4i}{-3+4i} = \dfrac{50(-3+4i)}{(-3-4i)(-3+4i)}$

$= \dfrac{-150 + 200i}{9 + 12i - 12i - 16i^2}$

$= \dfrac{-150 + 200i}{9 - 16i^2}$

$= \dfrac{-150 + 200i}{9 - 16(-1)}$

P 7 5 7 4 3 A 0 1 0 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 4 continued

$= \dfrac{-150+200i}{25}$

$= -\dfrac{150}{25} + \dfrac{200}{25}i$

$= -6+8i$

$-6+8i = kz$

$-6+8i = k(-3+4i)$

$-6+8i = -3k + 4ki$

equate real components and imaginary components:

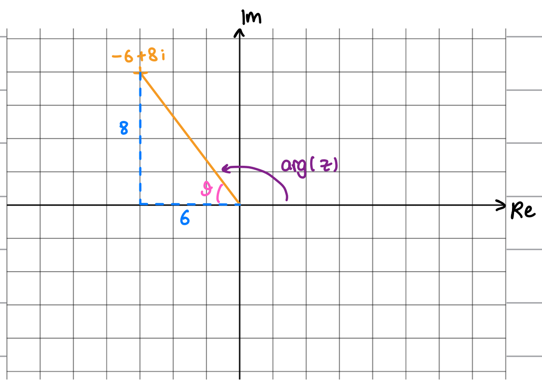
$-6 = -3k \qquad 8i = 4ki$

$2 = k \qquad 8 = 4k$

$2 = k$

$\boxed{\therefore k = 2}$

c. $\arg\left(\dfrac{50}{z^*}\right) = \arg(-6+8i)$ use part (b)



$\theta = \tan^{-1}\left(\frac{8}{6}\right)$

$\arg(z) = \pi - \tan^{-1}\left(\frac{8}{6}\right)$

$\arg(z) = 2.214297436^c$

$\boxed{\arg\left(\dfrac{50}{z^*}\right) = 2.21^c \ (3s.f.)}$

(Total for Question 4 is 8 marks)

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

5. The equation $5x^2 - 4x + 2 = 0$ has roots $\frac{1}{p}$ and $\frac{1}{q}$

(a) Without solving the equation,

(i) show that $pq = \frac{5}{2}$

(ii) determine the value of $p + q$

(4)

(b) Hence, without finding the values of p and q, determine a quadratic equation with roots

$$\frac{p}{p^2+1} \text{ and } \frac{q}{q^2+1}$$

giving your answer in the form $ax^2 + bx + c = 0$ where a, b and c are integers.

(5)

ai. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$x^2 - \left(\frac{1}{p} + \frac{1}{q}\right)x + \left(\frac{1}{p} \times \frac{1}{q}\right) = 0$

$x^2 - \left(\frac{p+q}{pq}\right)x + \left(\frac{1}{pq}\right) = 0$

$5x^2 - 4x + 2 = 0$

$x^2 - \frac{4}{5}x + \frac{2}{5} = 0$

$\frac{p+q}{pq} = \frac{4}{5}$

$\frac{1}{pq} = \frac{2}{5}$

$\frac{1}{pq} = \frac{2}{5}$

$5 = 2pq$

$\frac{5}{2} = pq$

$\boxed{\therefore pq = \frac{5}{2} \text{ // shown}}$

ii. $\frac{p+q}{pq} = \frac{4}{5}$

$\frac{p+q}{\left(\frac{5}{2}\right)} = \frac{4}{5}$ sub in pq value from part (ai)

$p + q = \frac{\cancel{4}^{2}}{\cancel{5}_{1}} \times \frac{\cancel{5}^{1}}{\cancel{2}_{1}}$

$\boxed{p + q = 2}$

P 7 5 7 4 3 A 0 1 2 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 5 continued

b. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

$$x^2 - \left(\frac{p}{p^2+1} + \frac{q}{q^2+1}\right)x + \left(\frac{p}{p^2+1} \times \frac{q}{q^2+1}\right) = 0$$

☆ $p + q = 2$

☆ $pq = \frac{5}{2}$

$$x^2 - \left(\frac{p(q^2+1) + q(p^2+1)}{(p^2+1)(q^2+1)}\right)x + \left(\frac{pq}{(p^2+1)(q^2+1)}\right) = 0$$

$$x^2 - \left(\frac{pq^2 + p + qp^2 + q}{p^2q^2 + p^2 + q^2 + 1}\right)x + \left(\frac{pq}{p^2q^2 + p^2 + q^2 + 1}\right) = 0$$

$$x^2 - \left(\frac{(p+q) + (pq^2 + qp^2)}{(pq)^2 + 1 + (p^2 + q^2)}\right)x + \left(\frac{pq}{(pq)^2 + 1 + (p^2+q^2)}\right) = 0$$

$$x^2 - \left(\frac{(p+q) + pq(q+p)}{(pq)^2 + 1 + (p+q)^2 - 2pq}\right)x + \left(\frac{pq}{(pq)^2 + 1 + (p+q)^2 - 2pq}\right) = 0$$

$$x^2 - \left(\frac{(2) + \left(\frac{5}{2}\right)(2)}{\left(\frac{5}{2}\right)^2 + 1 + (2)^2 - 2\left(\frac{5}{2}\right)}\right)x + \left(\frac{\left(\frac{5}{2}\right)}{\left(\frac{5}{2}\right)^2 + 1 + (2)^2 - 2\left(\frac{5}{2}\right)}\right) = 0$$

$$x^2 - \left(\frac{28}{25}\right)x + \left(\frac{2}{5}\right) = 0$$

$$x^2 - \frac{28}{25}x + \frac{2}{5} = 0$$

$$x^2 - \frac{28}{25}x + \frac{2}{5} = 0 \quad \times 25$$

$25x^2 - 28x + 10 = 0$ ← all integer coefficients

$\boxed{25x^2 - 28x + 10 = 0 \qquad a = 25,\ b = -28,\ c = 10}$

6. (a) Prove by induction that for $n \in \mathbb{Z}^+$

$$\begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}^n = \begin{pmatrix} 1 & (2^n - 1)r \\ 0 & 2^n \end{pmatrix}$$

where r is a constant.

(4)

$$\mathbf{M} = \begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix} \qquad \mathbf{N} = \begin{pmatrix} 1 & -2 \\ 0 & 2 \end{pmatrix}^4$$

The transformation represented by matrix **M** followed by the transformation represented by matrix **N** is represented by the matrix **B**

(b) (i) Determine **N** in the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c and d are integers.

(ii) Determine **B**

(3)

Hexagon S is transformed onto hexagon S' by matrix **B**

(c) Given that the area of S' is 720 square units, determine the area of S

(2)

a. 1) Base Case: Prove true for $n=1$

LHS: $\begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}^1 = \begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}$

RHS: $\begin{bmatrix} 1 & (2^1-1)r \\ 0 & 2^1 \end{bmatrix} = \begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}$

LHS = RHS $\therefore$ true for $n=1$

2) Assume true for $n=k$:

$\begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}^k = \begin{bmatrix} 1 & (2^k-1)r \\ 0 & 2^k \end{bmatrix}$

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 6 continued

(3) Prove true for $n = k+1$:

Thinking ahead, sub in $n = k+1$ into proof:

this is what we are trying to prove using (2):

$$\begin{pmatrix} 1 & r \\ 0 & 2 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & (2^{k+1}-1)r \\ 0 & 2^{k+1} \end{pmatrix}$$

Make a note so we know what we are working towards

$$\begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}^{k} \begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & (2^k-1)r \\ 0 & 2^k \end{bmatrix} \begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} (1)(1) + (2^k-1)r(0) & (1)(r) + (2^k-1)r(2) \\ (0)(1) + (2^k)(0) & (0)(r) + (2^k)(2) \end{bmatrix}$$

$$\begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & r + 2(2^k-1)r \\ 0 & 2(2^k) \end{bmatrix}$$

$$\begin{bmatrix} 1 & r \\ 0 & 2 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & (2^{k+1}-1)r \\ 0 & 2^{k+1} \end{bmatrix}$$

$\therefore$ true for $n = k+1$

(4) Conclusion

If this result is true for $n = k$, then it is true for $n = k+1$. As the result has been shown to be true for $n = 1$, then the result is true for all n.

bi. use part (a) – replace $r = -2$ and $n = 4$

$$\begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}^{4} = \begin{bmatrix} 1 & (2^4-1)(-2) \\ 0 & 2^4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -30 \\ 0 & 16 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}^{4} = \begin{bmatrix} 1 & -30 \\ 0 & 16 \end{bmatrix}}$$

Question 6 continued

ii. order: $B = NM$

$$B = \begin{bmatrix} 1 & -30 \\ 0 & 16 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} (1)(4) + (-30)(0) & (1)(0) + (-30)(5) \\ (0)(4) + (16)(0) & (0)(0) + (16)(5) \end{bmatrix}$$

$$\boxed{B = \begin{bmatrix} 4 & -150 \\ 0 & 80 \end{bmatrix}}$$

c. $|\det(B)|$ = area scale factor

$$X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$\det(X) = (a)(d) - (b)(c)$

$$B = \begin{bmatrix} 4 & -150 \\ 0 & 80 \end{bmatrix}$$

$\det(B) = (4)(80) - (-150)(8)$

$= 320$

$\det(B) = 320$

Area S.F. $= |\det(B)| = |320| = 320$

Area of $S \times 320 =$ Area of S'

$x \times 320 = 720$

$x = \frac{720}{320} = 9/4$

$\boxed{\text{Area of } S = 9/4 \text{ units}^2}$

P 7 5 7 4 3 A 0 1 8 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

7. **In this question use the standard results for summations.**

(a) Show that for all positive integers n

$$\sum_{r=1}^{n}(12r^2 + 2r - 3) = An^3 + Bn^2$$

where A and B are integers to be determined.

(4)

(b) Hence determine the value of n for which

$$\sum_{r=1}^{2n} r^3 - \sum_{r=1}^{n}(12r^2 + 2r - 3) = 270$$

(4)

a. $\sum_{r=1}^{n}(12r^2 + 2r - 3)$

split summation into 3

$= \sum_{r=1}^{n} 12r^2 + \sum_{r=1}^{n} 2r - \sum_{r=1}^{n} 3$

factor 12 out summation

factor 2 out summation

factor 3 out summation

$= 12\sum_{r=1}^{n} r^2 + 2\sum_{r=1}^{n} r - 3\sum_{r=1}^{n} 1$

Summations

$\sum_{r=1}^{n} 1 = n$

$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$

must learn!

$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$

$\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

in the formulae booklet

$= 12\left[\frac{n}{6}(n+1)(2n+1)\right] + 2\left[\frac{1}{2}n(n+1)\right] - 3[n]$

$= 2n(n+1)(2n+1) + n(n+1) - 3n$

factor out n

$= n[2(n+1)(2n+1) + (n+1) - 3]$

$= n[2(2n^2 + 3n + 1) + n + 1 - 3]$

$= n[4n^2 + 6n + 2 + n + 1 - 3]$

$= n[4n^2 + 7n]$

$= 4n^3 + 7n^2$

$\boxed{4n^3 + 7n^2 \quad A = 4,\ B = 7}$

P 7 5 7 4 3 A 0 2 0 2 8

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Question 7 continued

b. $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

$\therefore \sum_{r=1}^{2n} r^3 = \frac{1}{4}(2n)^2(2n+1)^2$

$\sum_{r=1}^{2n} r^3 - \underbrace{\sum_{r=1}^{n}(12r^2+2r-3)}_{\text{use ans from part (a)}} = 270$

$\frac{1}{4}(2n)^2(2n+1)^2 - (4n^3+7n^2) = 270$

$\cancel{\frac{1}{4}}(\cancel{4}n^2)(2n+1)^2 - 4n^3 - 7n^2 = 270$

$n^2(4n^2+4n+1) - 4n^3 - 7n^2 = 270$

factor out n^2

$n^2[4n^2+4n+1-4n-7] = 270$

$n^2[4n^2-6] = 270$

$4n^4 - 6n^2 - 270 = 0$

$2n^4 - 3n^2 - 135 = 0$ can alternatively solve via hidden quadratic

$(2n^2+15)(n^2-9) = 0$

$2n^2+15 = 0$ $\quad$ $n^2-9=0$

$n^2 = -\frac{15}{2}$ $\quad$ $n^2 = 9$

$n = \pm\sqrt{\frac{15}{2}}\,i$ $\quad$ $n = \pm 3$

$2n^4 - 3n^2 - 135 = 0$

$2(n^2)^2 - 3n^2 - 135 = 0$

let $u = n^2$

$2u^2 - 3u - 135 = 0$

$(2u+15)(u-9) = 0$

substitute $u = n^2$ again

$(2n^2+15)(n^2-9) = 0$ //

Q states n is a positive integer $\therefore n = 3$ only

$\boxed{n=3}$

(Total for Question 7 is 8 marks)

8. Prove by induction that for $n \in \mathbb{Z}^+$

$$f(n) = 7^{n-1} + 8^{2n+1}$$

is divisible by 57

(6)

(1) Base Case : check true for $n = 1$

$) = 7^{(1)-1} + 8^{2(1)+1}$

$= 513$

$= 9(57)$ ← divisible by 57

$\therefore$ true for $n = 1$

(2) Assume true for $n = k$:

$f(k) = 7^{k-1} + 8^{2k+1}$

(3) Inductive Step, prove true for $n = k+1$:

$f(k+1) = 7^{(k+1)-1} + 8^{2(k+1)+1}$

$= 7^{k} + 8^{2k+2+1}$

$= 7^{k} + 8^{2k+3}$

$f(k+1) - f(k) = (7^{k} + 8^{2k+3}) - (7^{k-1} + 8^{2k+1})$

$= [(7)^1(7)^{k-1} + (8)^2(8)^{2k+1}] - (7^{k-1} + 8^{2k+1})$

$= 7(7^{k-1}) + 64(8^{2k+1}) - (7^{k-1}) - (8^{2k+1})$

$= 6(7^{k-1}) + 63(8^{2k+1})$

$= 6[(7)^{k-1} + (8)^{2k+1}] + 57(8^{2k+1})$

$= 6f(k) + 57(8^{2k+1})$

$f(k+1) - f(k) = 6f(k) + 57(8^{2k+1})$

$f(k+1) = 7f(k) + 57(8^{2k+1})$

$\therefore f(k+1) = 7f(k) + 57(8^{2k+1})$ $\therefore$ true for $n = k+1$

P 7 5 7 4 3 A 0 2 2 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 8 continued

(4) Conclusion

If this result is true for $n=k$, then it is true for $n=k+1$. As the result has been shown to be true for $n=1$, then the result is true for all n.

(Total for Question 8 is 6 marks)

9. The rectangular hyperbola H has equation $xy = c^2$ where c is a positive constant.

The point $P\left(ct, \frac{c}{t}\right)$, where $t > 0$, lies on H

(a) Use calculus to show that an equation of the normal to H at P is

$$t^3x - ty = c(t^4 - 1)$$

(4)

The parabola C has equation $y^2 = 6x$

The normal to H at the point with coordinates (8, 2) meets C at the point Q where $y > 0$

(b) Determine the exact coordinates of Q

(4)

Given that

- the point R is the focus of C
- the line l is the directrix of C
- the line through Q and R meets l at the point S

(c) determine the exact length of QS

(5)

a. $xy = c^2$

$y = \frac{c^2}{x} = c^2x^{-1}$

$\frac{dy}{dx} = -c^2x^{-2} = \frac{-c^2}{x^2}$

Sub in point $P\left(ct, \frac{c}{t}\right)$:

$\left.\frac{dy}{dx}\right|_{(ct, \frac{c}{t})} \quad \frac{-c^2}{(ct)^2} = \frac{-c^2}{c^2t^2} = \frac{-1}{t^2}$

$M_{tangent} : -\frac{1}{t^2}$

$M_{normal} : t^2$

Set up line eq^n $y - y_1 = m(x - x_1)$ with $m = t^2$

$(x_1, y_1) = \left(ct, \frac{c}{t}\right)$

$y - \frac{c}{t} = t^2(x - ct)$

$t\left(y - \frac{c}{t}\right) = t\left(t^2(x - ct)\right)$

$ty - c = t^3(x - ct)$

$ty - c = t^3x - ct^4$

$ct^4 - c = t^3x - ty \Rightarrow \boxed{t^3x - ty = c(t^4 - 1) \text{ // shown}}$

P 7 5 7 4 3 A 0 2 4 2 8

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA DO NOT WRITE IN THIS AREA

Question 9 continued

b. (8,2) lies on hyperbola, H.

$\therefore \left(ct, \frac{c}{t}\right) = (8,2)$

$ct = 8$ $\quad\quad$ $\frac{c}{t} = 2$

$c = 2t$

Sub in $c = 2t$

$2t \times t = 8$

$2t^2 = 8$

$t^2 = 4$

$t = \pm\sqrt{4} = \pm 2$

However, Q states $t > 0$ $\quad \therefore t = +2$

Sub $t = 2$ back into either eqn to find c.

$c = 2t$

$c = 2(2) = 4$

$\therefore c = 4, t = 2$

Sub $c = 4, t = 2$ into normal line eqn

$2^3 x - 2y = 4(2^4 - 1)$

$8x - 2y = 60$

Solve $8x - 2y = 60$ and $y^2 = 6x$ simultaneously.

(1) $8x - 2y = 60$

(2) $y^2 = 6x$

(1) $8x - 2y = 60$ $\quad \div 2$

$4x - y = 30$

$y = 4x - 30$

sub $y = 4x - 30$ into (2)

$(4x - 30)^2 = 6x$

$16x^2 - 240x + 900 = 6x$

$16x^2 - 246x + 900 = 0$

Question 9 continued

$(8x-75)(2x-12)=0$

$x=\frac{75}{8}$ or $x=6$

sub into either
(1) or (2) to find y-coord.

$8x-2y=60$

$4x-y=30$

$y=4x-30$

$x=\frac{75}{8}, \quad y=4\left(\frac{75}{8}\right)-30=\frac{15}{2}$

$x=6, \quad y=4(6)-30=-6$

$\left(\frac{75}{8},\frac{15}{2}\right)$ or $\cancel{(6,-6)}$

Q states $y>0$

$\therefore$ must reject sol^n

$\boxed{Q\left(\frac{75}{8},\frac{15}{2}\right)}$

c. $y^2=6x$

$y^2=4\left(\frac{6}{4}\right)x \qquad a=\frac{6}{4}=\frac{3}{2}$

focus $R\left(\frac{3}{2},0\right)$

directrix line ℓ : $x=-\frac{3}{2}$

Conics

	Ellipse	Parabola	Hyperbola	Rectangular Hyperbola
Standard Form	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$	$y^2=4ax$	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$	$xy=c^2$
Parametric Form	$(a\cos\theta, b\sin\theta)$	$(at^2, 2at)$	$(a\sec\theta, b\tan\theta)$ $(\pm a\cosh\theta, b\sinh\theta)$	$\left(ct, \frac{c}{t}\right)$
Eccentricity	$e<1$ $b^2=a^2(1-e^2)$	$e=1$	$e>1$ $b^2=a^2(e^2-1)$	$e=\sqrt{2}$
Foci	$(\pm ae, 0)$	$(a, 0)$	$(\pm ae, 0)$	$(\pm\sqrt{2}c, \pm\sqrt{2}c)$
Directrices	$x=\pm\frac{a}{e}$	$x=-a$	$x=\pm\frac{a}{e}$	$x+y=\pm\sqrt{2}c$
Asymptotes	none	none	$\frac{x}{a}=\pm\frac{y}{b}$	$x=0, y=0$

(1) find line eq^n between Q and R.

$M_{QR}=\dfrac{\frac{15}{2}-0}{\frac{75}{8}-\frac{3}{2}}=\dfrac{20}{21}$

Set up line eq^n $y-y_1=m(x-x_1)$ with $m=\frac{20}{21}$

$(x_1,y_1)=\left(\frac{3}{2},0\right)$

$y-0=\frac{20}{21}\left(x-\frac{3}{2}\right)$

$y=\frac{20}{21}x-\frac{10}{7}$ ★

P 7 5 7 4 3 A 0 2 6 2 8

DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA
DO NOT WRITE IN THIS AREA

Question 9 continued

Solve line ℓ and line through Q and R simultaneously to find point S:

(1) $x = -\frac{3}{2}$

(2) $y = \frac{20}{21}x - \frac{10}{7}$

Sub $x = -3/2$ into (2) and solve for y

$y = \frac{20}{21}\left(-\frac{3}{2}\right) - \frac{10}{7}$

$y = -\frac{20}{7}$

$\therefore S\left(-\frac{3}{2}, -\frac{20}{7}\right)$

(3) find $\| QS \|$

$\sqrt{\left(\frac{75}{8} - -\frac{3}{2}\right)^2 + \left(\frac{15}{2} - -\frac{20}{7}\right)^2}$

Distance between 2 points:

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ for points (x_1, y_1) & (x_2, y_2)

$= \frac{841}{56}$

$\boxed{\| QS \| = \frac{841}{56} \text{ units}}$